lectures on functional analysis (1)

first course

The fourth stage

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<u>المحاضرة الاولى</u>

① Functional Analysis 2021

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References

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1. Topological spaces

Definition:

let τ be a collection of sub sets of a set X. we Say that τ is topology on X if the following Conditions are holds:

i. $\phi \in \tau$ and $X \in \tau$

ii. Union of every members in τ is also in τ.

iii. Intersection of any finite members in τ is also in τ.

Remarks:

1. The set X together of τ are Called a topological Space and is denoted by (X, τ).

2. A members of τ are called open sets i.e. a sub set A of X is called an open set in X, if $A \in \tau$ and we say that A is closed set in X if A^c is open set in X.

3. A neighborhood of a point $x \in X$ is any open set contains x.

4. The interior of A is the union of all open sets in X that are sub sets of A.

5. The closure $\bar{A}$ of A is the intersection of all closed sets in X that contain A.

6. A function $f: \overset{T.S.}{X} \longrightarrow \overset{T.S.}{Y}$ is cont^s. at $x \in X$ if $\forall$ neighborhood U of $f(x)$ in Y $\exists$ neighborhood V of x in X s.t. $f(V) \subset U$.

linear spaces:

The letter $\mathbb{R}$ and $\mathbb{C}$ will always denote the field of real numbers and the field of complex numbers respectively. Hence F stand either $\mathbb{R}$ or $\mathbb{C}$.

Definition:

A linear space over F is a set X, whose elements are called vectors and in which two operations, addition $+: X \times X \longrightarrow X$ and scalar multiplication $\cdot: F \times X \longrightarrow X$ such that

1. $(X, +)$ abelian group.
2. $\alpha \cdot x \in X$ $\forall \alpha \in F$ and $x \in X$.
3. $\alpha \cdot (x + y) = \alpha \cdot x + \alpha \cdot y$ $\forall \alpha \in F$ & $x, y \in X$.
4. $(\alpha + \beta) \cdot x = \alpha \cdot x + \beta . x$ $\forall \alpha, \beta \in F$ & $x \in X$.
5. $(\alpha \cdot \beta) x = \alpha \cdot (\beta \cdot x)$ $\forall \alpha, \beta \in F$ & $x \in X$.
6. $1 \cdot x = x$ $\forall x \in X$ and 1 is the unity element of the field F.

Definition:

Let A be a subset of linear space X over F. We say that A is

1. Symmetric if $-A = A$, so that $A \cap (-A)$ is symm. for any subset A of X.
2. Balanced if $\alpha A \subset A$ for every $\alpha \in F$ with $|\lambda| \leq 1$.
3. Absorbing if for every $x \in X$, $\exists \lambda \in F$, $\lambda > 0 \ni x \in \lambda A$.

Theorem :

④

let A and B be subsets of a linear space X.

1. If A are balanced sets in linear space X and $\alpha \in F \ni |\alpha| = 1$, then $\lambda A = A$, and every balanced set is symmetric.

2. If A and B are balanced sets in a linear space, then $A \cap B$, $A \cup B$ and $A+B$ are also balanced in X.

Proof: (1) Since A is balanced $\Rightarrow \lambda A \subset A$ $\forall \lambda \in F$ with $|\lambda| \leq 1$.

$\Rightarrow \lambda A \subset A$ when $|\lambda| = 1$, we need to show that $A \subset \lambda A$.

let $x \in A$, since $|\lambda| \neq 0$, we set $\alpha = \frac{1}{\lambda}$
$\Rightarrow |\alpha| = 1$

Since A is balanced set $\Rightarrow \alpha A \subset A \Rightarrow \alpha x \in A$
$\Rightarrow \lambda(\alpha x) \in \lambda A \Rightarrow x \in \lambda A \Rightarrow A \subset \lambda A$
$\Rightarrow \lambda A = A$.

Now, we show that A is symmetric, put $\lambda = -1$
$\Rightarrow |\lambda| = 1$
we have $\lambda A = A \Rightarrow -A = A \Rightarrow A$ is symmetric.

(2) let $\lambda \in F$ with $|\lambda| \leq 1$, from (2) A ∧ B are balanced sets $\Rightarrow \lambda A \subset A$ ∧ $\lambda B \subset B$

i. let $x \in \lambda(A \cap B) \Rightarrow x = \lambda y \ni y \in A \cap B$
$\Rightarrow y \in A$ ∧ $y \in B$

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$\Rightarrow \quad x \in \lambda A$ or $x \in \lambda B$ because
$\lambda y \in \lambda A$ or $\lambda y \in \lambda B$

$\Rightarrow \quad x \in A$ or $x \in B \Rightarrow x \in A \cap B$

$\Rightarrow \quad \lambda(A \cap B) \subset A \cap B \Rightarrow A \cap B$ balanced set.

ii. let $x \in \lambda(A+B) \Rightarrow x = \lambda(a+b) \Rightarrow$
$a \in A$ or $b \in B$

$\Rightarrow \quad x = \lambda a + \lambda b \Rightarrow \lambda a \in A$ because $\lambda A \subset A$

also $\lambda b \in B$ because $\lambda B \subset B$

$\Rightarrow \quad \lambda(a+b) \in A+B \Rightarrow x \in A+B$

$\Rightarrow \quad \lambda(A+B) \subset A+B$

$\Rightarrow \quad A+B$ is balanced set.

Definition:

let M be a subset of a linear space X. we say that M is a subspace of X if M itself is a linear space.

It is clear to show: A non-empty sub set M of a linear space X is subspace of X iff

1. $x+y \in M \quad \forall x,y \in M$ 2. $\alpha x \in M \quad \forall x \in M$ & $\alpha \in F$

or equivaalently, $\alpha x + \beta y \in M \quad \forall \alpha, \beta \in F$ & $x, y \in M$.

Remark:

Every linear space X has at Least two trivial subspaces namely X itself and the zero subspace $\{0\}$

Definition:

let M_1 and M_2 be two subspaces of a linear space X. Then

1. $M_1 \cap M_2$ is a subspace of X

2. $M_1 \cup M_2$ is a subspace of X iff $M_1 \subseteq M_2$ or $M_2 \subseteq M_1$.

3. $M_1 + M_2$ is a subspace of X and $M_1 \subseteq M_1 + M_2$, $M_2 \subseteq M_1 + M_2$.

Proof: (1) Since $0 \in M_1$ and $0 \in M_2 \Rightarrow 0 \in M_1 \cap M_2$

$\Rightarrow M_1 \cap M_2 \neq \emptyset$

let $x, y \in M_1 \cap M_2$ and $\alpha, \beta \in F \Rightarrow x, y \in M_1$ and $x, y \in M_2$

Since M_1 and M_2 are subspace of linear space X.

$\Rightarrow \alpha x + \beta y \in M_1$ and $\alpha x + \beta y \in M_2$

$\Rightarrow \alpha x + \beta y \in M_1 \cap M_2 \Rightarrow M_1 \cap M_2$ subspace of X.

Definition:

let X be a linear space over field F and let $x_1, x_2, \dots, x_n \in X$. we say that x is linear combination of $x_1, x_2, \dots, x_n$ if $x = \sum_{i=1}^{n} \alpha_i x_i$ where $\alpha_i \in F$, $1 \leq i \leq n$

Note: let A be a non-empty subset of X. The set of all linear combination of finite elements of A denoted by $[A]$ i.e. $[A] = \{ x = \sum_{i=1}^{n} \alpha_i x_i ,\ x_i \in A,\ \alpha_i \in F,\ 1 \leq i \leq n \}$

Lemma:

let A be a non-empty sub set of linear space X. Then $[A]$ is smallest subspace of X which contains A is called the subspace spanned (or generated) by A.

proof: Now, to prove $A \subseteq [A]$

let $x \in [A]$, since $1 \in F \Rightarrow 1 \cdot x \in [A] \Rightarrow x \in [A]$

$\Rightarrow A \subseteq [A]$

we need to show that $[A]$ is a subspace of X.

Since $A \neq \phi$ and $A \subseteq [A] \Rightarrow [A] \neq \phi$

let $x, y \in [A]$ and $\alpha, \beta \in F$

$\Rightarrow x = \sum_{i=1}^{n} \alpha_i x_i$ and $y = \sum_{j=1}^{m} \beta_j y_j$ where $\alpha_i, \beta_j \in F$

$1 \leq i \leq n$ and $1 \leq j \leq m$

$$\alpha x + \beta y = \alpha \left(\sum_{i=1}^{n} \alpha_i x_i \right) + \beta \left(\sum_{j=1}^{m} \beta_j y_j \right)$$

$$= (\alpha \alpha_1) x_1 + \cdots + (\alpha \alpha_n) x_n + (\beta \beta_1) y_1 + \cdots + (\beta \beta_m) y_m$$

$\Rightarrow \alpha x + \beta y$ is a linear combination of finite elements of $A \Rightarrow \alpha x + \beta y \in [A]$.

Consider M subspace of X such that $A \subseteq M$. T.P. $[A] \subseteq M$.

let $x \in [A] \Rightarrow x = \sum_{i=1}^{n} \alpha_i x_i$, where $\alpha_i \in F$ and $x_i \in A$ for $1 \leq i \leq n$

Since $A \subseteq M \Rightarrow x_i \in M$

Since M is a subspace of $X \Rightarrow x = \sum_{i=1}^{n} \alpha_i x_i \in M$

$\Rightarrow [A] \subseteq M$.

Remarks: If A is a subset of L.S. X, then

1. $[A]$ = intersection of all subspaces of X which containing A.

2. A is a subspace iff $A = [A]$.

3. If $A = \{x_0\}$, we write $[x_0]$ instate of $[\{x_0\}]$

so that, $[x_0] = \{x = \lambda x_0 : \lambda \in F\}$

4. If $x_0 \notin A$, then $[A \cup \{x_0\}]$ is subspace generated by $A \cup \{x_0\}$ and

$[A \cup \{x_0\}] = \{x = a + \lambda x_0 ; a \in A \text{ \& } \lambda \in F\}$.

Theorem:

let X be topological linear space and let $\lambda \in F$, $A, B \subseteq X$. Then

1. $\bar{A} = \cap \{A + V\}$, where V runs through all neighbor. of 0

2. $\overline{\lambda A} = \lambda \bar{A}$.

3. $\bar{A} + \bar{B} \subset \overline{A + B}$

4. If A is a subspace of X, so is $\bar{A}$.

5. If A is a balanced subset of X, so is $\bar{A}$.

6. If A is a balanced subset of X and $0 \in \text{int}(A)$, then $\text{int}(A)$ is balanced.

proof: (1) let $x \in \bar{A} \Rightarrow$ for every neighborhood V of 0,

Then $(x + V) \cap A \neq \phi$

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$\Rightarrow \exists\, y \in (x+V) \cap A \Rightarrow y \in x+V$ and $y \in A$

Since $y \in x+V \Rightarrow y = x+b$ such that $b \in V$

$\Rightarrow x = y - b$ such that $b \in V$, $y \in A$

$\Rightarrow x \in A - V$

Hence $x \in A + V$ is a neighborhood of 0.

(3) let $x \in \bar{A} + \bar{B} \Rightarrow x = a + b \ni a \in \bar{A}$ and $b \in \bar{B}$

and let W be a neighborhood of $a+b$

Since the function $+ : X \times X \longrightarrow X$ is continuous, there are neighborhoods V_a and V_b such that

$V_a + V_b \subset W$

Since $a \in \bar{A} \Rightarrow V_a \cap A \neq \Phi \Rightarrow \exists\, y \in V_a \cap A$

Since $b \in \bar{B} \Rightarrow V_b \cap B \neq \Phi \Rightarrow z \in V_b \cap B$

$y + z \in V_a + V_b \subset W \Rightarrow y + z \in W$

$\Rightarrow y + z \in W \cap (A + B)$

$\Rightarrow W \cap (A+B) \neq \Phi$ for each a neighborhood W of $a+b \Rightarrow a + b \in \overline{A+B}$

$\Rightarrow \bar{A} + \bar{B} \subset \overline{A+B}$.

(4) Let $\alpha, \beta \in F$, we shall to show that $\alpha\bar{A} + \beta\bar{A} \subset \bar{A}$

Since A is subspace of X, then $\alpha A + \beta A \subset A$

$\Rightarrow \overline{\alpha A + \beta A} \subset \bar{A}$

If $\alpha = 0 \Rightarrow \alpha A = \{0\}$ and if $\alpha \neq 0 \Rightarrow \overline{\alpha A} = \alpha \bar{A}$

$\Rightarrow \alpha\bar{A} + \beta\bar{A} = \overline{\alpha A} + \overline{\beta A} \subset \overline{\alpha A + \beta A} \subset \bar{A}$

$\Rightarrow \bar{A}$ is subspace of X.

(5) let $\lambda \in F$ such that $|\lambda| \leq 1$

since A is a balanced $\lambda A \subset A \Rightarrow \overline{\lambda A} \subset \bar{A}$

$\Rightarrow \overline{\lambda A} = \lambda \bar{A} \subset \bar{A} \Rightarrow \bar{A}$ is balanced of X.

(6) If $0 < |\lambda| \leq 1 \Rightarrow \lambda \operatorname{int}(A) = \operatorname{int}(\lambda A) \subset \lambda A \subset A$

since $\overline{\lambda A} = \lambda \bar{A}$, λA° is open set and

$\lambda \operatorname{int}(A) \subset A$, then $\lambda \operatorname{int}(A) \subset \operatorname{int}(A)$.

because $\operatorname{int}(A)$ is greatest open set which contain A.

If $\lambda = 0$, since $0 \in \operatorname{int}(A) \Rightarrow \lambda \operatorname{int}(A) = \{0\}$

$\Rightarrow \{0\} \subset \operatorname{int}(A) \Rightarrow \operatorname{int}(A)$ is balanced.

Definition:

A subset A of a linear space X. We say that A is Convex if $\lambda x + (1-\lambda) y \in A$, whenever $x, y \in A$, $0 \leq \lambda \leq 1$

or equivalently if $\lambda A + (1-\lambda) A \subset A$, $\forall$ $0 < \lambda \leq 1$.

(Every open set in X is a union of Convex open sets).

Example: • The empty set and the set consisting of one point are Convex.

• Every subspace of a linear space is Convex, but the Converse is not true.

Remark:

If A is subset of a linear space X over F, then

$(\alpha + \beta) A \subset \alpha A + \beta A$

Indeed: If $x \in (\alpha + \beta) A$, then $x = (\alpha + \beta) a$, $a \in A$

$\Rightarrow x = \alpha a + \beta a \in \alpha A + \beta A$

In general $\alpha A + \beta A \not\subset (\alpha + \beta) A$.

Theorem:

If A is a subset of a linear space X, then A is Convex if and only if $(\alpha+\beta)A = \alpha A + \beta A$ for $\alpha, \beta \in \mathbb{R}^+$.

Proof: suppose that Convex and to prove $(\alpha+\beta)A = \alpha A + \beta A$ for $\alpha, \beta \in \mathbb{R}^+$.

let $x \in \alpha A + \beta A \Rightarrow x = \alpha a + \beta b$, where $a, b \in A$

$$x = (\alpha+\beta)\left(\frac{\alpha}{\alpha+\beta} a + \frac{\beta}{\alpha+\beta} b\right)$$

Put $\lambda = \frac{\alpha}{\alpha+\beta} \Rightarrow 1-\lambda = \frac{\beta}{\alpha+\beta}$

Since $\alpha, \beta \in \mathbb{R}^+ \Rightarrow \lambda \geq 0$

Since $\alpha \leq \alpha+\beta \Rightarrow \lambda \leq 1 \Rightarrow 0 \leq \lambda \leq 1$

Since A is Convex, then $\lambda a + (1-\lambda) b \in A$

$\Rightarrow \frac{\alpha}{\alpha+\beta} a + \frac{\beta}{\alpha+\beta} b \in A$

$\Rightarrow \alpha a + \beta b \in (\alpha+\beta) A$

$\Rightarrow$

$x \in (\alpha+\beta) A \Rightarrow \alpha A + \beta A \subseteq (\alpha+\beta) A$

we have $(\alpha+\beta) A \subseteq \alpha A + \beta A$

Thus $(\alpha+\beta) A = \alpha A + \beta A$.

Conversely: let $(\alpha+\beta) A = \alpha A + \beta A$ for $\alpha, \beta \in \mathbb{R}^+$

let $0 \leq \lambda \leq 1 \Rightarrow 1-\lambda \geq 0$. Then

$\lambda A + (1-\lambda) A = (\lambda + (1-\lambda)) A = A$

$\Rightarrow \lambda A + (1-\lambda) A \subseteq A \Rightarrow A$ is Convex.

$\square$

Theorem:

let A and B be subsets ~~space~~ of linear space X.
If A & B are Convex sets in X and $\lambda \in F$ (field),
then $A \cap B$, αA, $A+B$ are also convex sets in X.

Proof: 1. let $x, y \in A \cap B$ and $0 \le \lambda \le 1$

$\Rightarrow$ $x, y \in A$ & $x, y \in B$

Since A & B are Convex, then

$\lambda x + (1-\lambda) y \in A$ & $\lambda x + (1-\lambda) y \in B$

$\Rightarrow$ $\lambda x + (1-\lambda) B \in A \cap B \Rightarrow A \cap B$ is Convex.

2. let $x, y \in \alpha A$ and $0 \le \lambda \le 1 \Rightarrow \left.\begin{array}{l} x = \alpha z \\ y = \alpha w \end{array}\right\} z, w \in A$

Since A is Convex $\Rightarrow \lambda z + (1-\lambda) w \in A$

$\Rightarrow \alpha(\lambda z + (1-\lambda) w) \in \alpha A$

$\Rightarrow \lambda(\alpha z) + (1-\lambda)\alpha w \in \alpha A$

$\Rightarrow \lambda x + (1-\lambda) y \in \alpha A$

$\Rightarrow$ αA is Convex.

3. let $x, y \in A+B$ and $0 \le \lambda \le 1$.

$x = a_1 + b_1$ & $y = a_2 + b_2$ & $a_1, a_2 \in A$, $b_1, b_2 \in B$

Since A & B are Convex, then $\lambda a_1 + (1-\lambda) a_2 \in A$
$\lambda b_1 + (1-\lambda) b_2 \in B$

$\Rightarrow \lambda(a_1 + b_1) + (1-\lambda)(a_2 + b_2) \in A + B$

$\Rightarrow \lambda x + (1-\lambda) y \in A + B$

$\Rightarrow$ $A + B$ Convex.

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linear Functional

Def.: let X and Y be linear spaces. A function $f: X \longrightarrow Y$ is called a linear if

$$f(\alpha x + \beta y) = \alpha f(x) + \beta f(y), \ \forall x, y \in X, \alpha, \beta \in F.$$

Remarks:

1. A function between two linear space is called an operator or transformation and it is linear if satisfies the above Condition.

2. Kernel (or null space) of a linear function $f: X \longrightarrow Y$ denoted by $ker(f)$ or $N(f)$ and defined by

$$N(f) = \{x \in X ; f(x) = 0\} = f^{-1}(\{0\})$$

3. linear function of a linear space X into its field F is called linear functional on X.

4. let $L(X, Y)$ denoted the set of all linear functions from linear space X into linear space Y. Then $L(X, Y)$ is a vector space under the following addition and scalar multiplication

i. for $f, g \in L(X, Y)$, $(f+g)(x) = f(x) + g(x)$

2. for $f \in L(X, Y)$ and $\alpha \in F$

$$(\alpha f)(x) = \alpha f(x).$$

If $\mathcal{Y} = \mathcal{X}$, we write $L(\mathcal{X})$ instead of $L(\mathcal{X}, \mathcal{X})$.

The space of all linear functionals defined on a linear space $\mathcal{X}$ is called the algebraic dual space and denoted by $\mathcal{X}'$ i.e. $\mathcal{X}' = L(\mathcal{X}, F)$.

5. We say that $\mathcal{X}, \mathcal{Y}$ are linear isomorphic (we write $\mathcal{X} \cong \mathcal{Y}$), the there is a bijection linear function $f: \mathcal{X} \to \mathcal{Y}$ such function is called linear isomorphism.

Theorem:

let $f: \mathcal{X} \to \mathcal{Y}$ be a linear function

1. $f(0) = 0$

2. $f(-x) = -f(x) \quad \forall x \in \mathcal{X}$

3. $f(x-y) = f(x) - f(y) \quad \forall x, y \in \mathcal{X}$

4. $f(\sum_{i=1}^{n} \alpha_i x_i) = \sum_{i=1}^{n} \alpha_i f(x_i) \quad \forall x_i \in \mathcal{X}, \alpha_i \in F$, $1 \le i \le n$.

5. If A is subspace (or Convex, balanced) in $\mathcal{X}$, the same is true $f(A)$.

6. If B is subspace (or Convex, balanced) in $\mathcal{Y}$, the same is true $f^{-1}(B)$.

7. The null space of f is linear space.

8. $\mathcal{N}(f) = \{0\} \Leftrightarrow f$ is injective.

Metric Linear spaces

Definition: let X be a non-empty set, $\mathbb{R}$ be a set of real numbers. A function $f := d : X \times X \longrightarrow \mathbb{R}$ is called metric function, if satisfies the following conditions:

1. $d(x,y) \geq 0 \quad \forall\, x,y \in X$
2. $d(x,y) = 0 \quad \text{iff } x = y, \quad \forall\, x,y \in X$
3. $d(x,y) = d(y,x) \quad \forall\, x,y \in X$
4. $d(x,y) \leq d(x,z) + d(z,y), \quad \forall\, x,y,z \in X$

A metric space is an order pair (X,d), where X be a non-empty set and d is metric function on X. Also the elements of X is called points and $d(x,y)$ is called the distance between x and y.

Remarks:

1. Usually, only three conditions are used to define a distance function. Indeed the first of these conditions is property that follows from the other three, since

$$2\, d(x,y) = d(x,y) + d(x,y) = d(x,y) + d(y,x)$$
$$\geq d(x,x) = 0$$

2. If all these conditions hold for (2) we only have $d(x,x) = 0$, then d is a pseudo metric.

We then call (X, d) a pseudo metric space. 16

3. Subspaces of a metric space are subsets whose metric is obtained by restricting the metric on the whole space.

A metric subspace (Y, d_Y) of metric space (X, d) consists of a subset $Y \subset X$ whose metric $d_Y : Y \times Y \to \mathbb{R}$ is restriction of d to Y, i.e. $d_Y(x, y) = d(x, y) \ \forall \ x, y \in Y$.

4. Many different metrics can be defined on the same set X, but if the metric on X is clear from the context, we refer to X as a metric space.

Examples:

1. The function $d_u : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ which defined by $d_u(x, y) = |x - y|$, $\forall \ x, y \in \mathbb{R}$ is metric function and hence $(\mathbb{R}, d_u)$ is metric space and this metric is called usual metric space.

2. The function $d : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ which defined by $d(x, y) = |x - y| + 1$, $\forall \ x, y \in X = \mathbb{R}$ is not a metric function.

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3. let X be a non-empty set. The function $d: X \times X \longrightarrow \mathbb{R}$ which defined by

$$d(x,y) = \begin{cases} 0 & , x = y \\ 1 & , x \neq y \end{cases} \quad \forall \ x,y \in X$$

is metric function and hence (X, d) is metric space and this metric is called dis~~tance~~crete metric space.

4. Euclidean spaces

i. Then function $d : \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}$ which define by

$$d(x,y) = \left(\sum_{i=1}^{n} (x_i - y_i)^2 \right)^{1/2} \quad \forall \ x,y \in \mathbb{R}^n$$

where $x = (x_1, x_2, \ldots, x_n)$ & $y = (y_1, y_2, \ldots, y_n)$ is metric function. Thus $(\mathbb{R}^n, d)$ is metric space.

ii. The function $d : \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}$ which defined by

$$d(x,y) = \sum_{i=1}^{n} | x_i - y_i | \quad , \forall \ x, y \in \mathbb{R}^n$$

where $x = (x_1, x_2, \ldots, x_n)$ & $y = (y_1, y_2, \ldots, y_n)$

is metric function. Thus $(\mathbb{R}^n, d)$ is metric space.

iii Also The function $d : \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}$ which defin by $d(x,y) = \max \{ | x_i - y_i | , 1 \leq i \leq n \}$ is metr

5. let (X, d_1) and (Y, d_2) be two metric spaces we define : $d((x_1, y_1), (x_2, y_2)) = \max \{ d_1(x_1, x_2), d_2(y_1, y_2) \}$

for all $(x_1, y_1), (x_2, y_2) \in X \times Y$. Then d is a metric on $X \times Y$ and $(X \times Y, d)$ is called the product of the metric spaces (X, d_1) and (Y, d_2). (18)

<u>Solution:</u> 1. let $(x_1, y_1), (x_2, y_2) \in X \times Y$ for $x_1, x_2 \in X$ & $y_1, y_2 \in Y$

$\Rightarrow d_1(x_1, x_2) \geq 0$ & $d_2(y_1, y_2) \geq 0$ because d_1 & d_2 are metric functions

$\Rightarrow \max\{d_1(x_1, x_2), d_2(y_1, y_2)\} \geq 0$

$\Rightarrow d((x_1, y_1), (x_2, y_2)) \geq 0$

2. let $(x_1, y_1), (x_2, y_2) \in X \times Y$ &

$d((x_1, y_1), (x_2, y_2)) = 0 \Leftrightarrow \max\{d_1(x_1, x_2), d_2(y_1, y_2)\} = 0$

$\Leftrightarrow d_1(x_1, x_2) = 0$ & $d_2(y_1, y_2) = 0$

$\Leftrightarrow x_1 = x_2$ & $y_1 = y_2$

$\Leftrightarrow (x_1, y_1) = (x_2, y_2)$

3. Let $(x_1, y_1), (x_2, y_2) \in X \times Y$

$$\begin{aligned} d((x_1, y_1), (x_2, y_2)) &= \max\{d_1(x_1, x_2), d_2(y_1, y_2)\} \\ &= \max\{d_2(y_1, y_2), d_1(x_1, x_2)\} \\ &= \max\{d_2(y_2, y_1), d_1(x_2, x_1)\} \\ &= d((x_2, y_2), (x_1, y_1)) \end{aligned}$$

4. let $(x_1, y_1), (x_2, y_2), (x_3, y_3) \in X \times Y$

$$\begin{aligned} d((x_1, y_1), (x_2, y_2)) &= \max\{d_1(x_1, x_2), d_2(y_1, y_2)\} \\ &\leq \max\{d_1(x_1, x_3) + d_1(x_3, x_2), d_2(y_1, y_3) + d_2(y_3, y_2)\} \end{aligned}$$

$\leq \max\{d_1(x_1,x_3), d_2(y_1,y_3)\} + \max\{d_1(x_3,x_2), d_2(y_3,y_2)\}$

$= d((x_1,y_1),(x_3,y_3)) + d((x_3,y_3),(x_2,y_2))$.

Theorem:

let (X,d) be pseudo-metric space. Define a relation $\sim$ on X by setting $x \sim y$ iff $d(x,y)=0$. Then

1. $\sim$ is an equivalence relation on X.
2. If $[x]$ is an equivalence class containing x and $A = \{[x] : x \in X\}$, then the function $d^*: A \times A \to \mathbb{R}$, defined by $d^*([x],[y]) = d(x,y)$, is a metric hence (A, d^*) is metric space.

Proof: (1)

Reflexivity, since $d(x,x) = 0$, $\forall x \in X \Rightarrow x \sim x$

Symmetric, we have $x \sim y \Rightarrow d(x,y) = 0 \Rightarrow$
$d(y,x) = 0 \Rightarrow y \sim x$.

Transitive, let $x \sim y$ and $y \sim z$. Then
$d(x,y) = 0$ & $d(y,z) = 0$

since $d(x,z) \leq d(x,y) + d(y,z) = 0$, but $d(x,z) \geq 0$

$\Rightarrow d(x,z) = 0 \Rightarrow x \sim z$.

Thus $\sim$ is equivalence relation on X.

(2) If $a \in [x]$ and $b \in [y]$, then $d(x,a) = 0$ and $d(y,b) = 0 \Rightarrow a \sim x$, $b \sim y$

since $| d(x,y) - d(a,b) | \leq d(x,a) + d(y,b)$ (20)

$\Rightarrow | d(x,y) - d(a,b) | \leq 0$

since the absolute value cannot be nagative, we must have $| d(x,y) - d(a,b)| = 0$.

Which implies that $d(x,y) - d(a,b) = 0$

$\Rightarrow d(x,y) = d(a,b)$. Hence d^* is well defined.

Finally we show that d^* is actually a metric on A.

1. since $d(x,y) \geq 0$ for all $x,y \in X$,

$d^*([x],[y]) \geq 0$ forall $[x],[y] \in A$.

2. let $x,y \in X$, $[x],[y] \in A$;

$d^*([x],[y]) = 0 \Leftrightarrow d(x,y) = 0 \Leftrightarrow x \sim y$

$\Leftrightarrow [x] = [y]$

3. let $x,y \in X$, $[x],[y] \in A$

$d^*([x],[y]) = d(x,y) = d(y,x) = d^*([y],[x])$.

4. let $x,y,z \in X$, $[x],[y],[z] \in A$

$d^*([x],[y]) = d(x,y) \leq d(x,z) + d(z,y)$

$= d^*([x],[z]) + d^*([z],[y])$.

$\Rightarrow (A, d^*)$ is a metric space.

$\square$

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Def.:

let (X, d) be a metric space and $A, B \subseteq X$:

i. The diameter of A is denoted by $\delta(A)$ and defined by $\delta(A) = \sup\{d(x,y) : x, y \in A\}$.

ii. The distance between a point $p \in X$ and A is denoted by $d(p, A)$ and defined by $d(p,A) = \inf\{d(p,x) ; x \in A\}$.

iii. The distance between A and B is denoted by $d(A, B)$ and defined by $d(A,B) = \inf\{d(x,y) ; x \in A, y \in B\}$.

Remarks:

i. $\delta(A) \geq 0$, $\delta(\phi) = 0$, $d(p,A) \geq 0$, $d(p, \phi) = \infty$, $d(A, B) \geq 0$ and $d(\phi, B) = \infty$.

ii. If $p \in A$, then $d(p, A) = 0$.

iii. If A, B are non-empty subsets of X such that $A \cap B \neq \phi$, then $d(A, B) = 0$ but the converse need not true.

Def.: let (X, d) be a metric space, r real number with $r > 0$. Then open ball $B_r(x_0)$ in X of center x_0 and radius r is the set of all points whose distance from x_0 is less than r i.e.

$$B_r(x_0) = \{x \in X ; d(x, x_0) < r\}.$$

The closed ball $\overline{B_r}(x_0)$ in X of center x_0 and radius r is the set of all points whose distance from x_0 is less than or equal r i.e. $\overline{B}_r(x_0)=\{x\in X, d(x,x_0)\leq r\}$. (22)

The sphere is the set of all points whose distance from the center x is equal r.

Remark:

Every open ball and closed ball are non-empty sets.

Example: 1. let $(\mathbb{R}, d)$ be a usual metric space and $x_0\in\mathbb{R}$, $r>0$,

$$B_r(x_0)=\{x\in\mathbb{R}, d(x,x_0)<r\}=\{x\in\mathbb{R}, |x-x_0|<r\}$$
$$=\{x\in\mathbb{R}, -r<x-x_0<r\}=\{x\in\mathbb{R}, x_0-r<x<x_0+r\}$$
$$=\{x\in\mathbb{R}, (x_0-r, x_0+r)=(a,b)=A$$

2. let (X,d) be a discrete metric space and $x_0\in X$, $r>0$.

i. If $r>1$, then $B_r(x_0)=X$

let $x\in X$, since $d(x,x_0)\begin{cases}0, & x=x_0\\ 1, & x\neq x_0\end{cases}\Rightarrow d(x,x_0)<r$

$\Rightarrow x\in B_r(x_0)\Rightarrow X\subset B_r(x_0)$, we have $B_r(x_0)\subset X$

$\Rightarrow X=B_r(x_0)$.

ii. If $r\leq 1$, then $B_r(x_0)=\{x_0\}$

let $x\in X \ni x\neq x_0\Rightarrow d(x,x_0)=1\Rightarrow d(x,x_0)\geq r$

$\Rightarrow x\notin B_r(x_0)\ \forall x\neq x_0$, since $x_0\in B_r(x_0)$

$\Rightarrow B_r(x_0)=\{x_0\}$.

Def.: let (X, d) be a metric space, set $A \subset X$ is called bounded if there exist $x_0 \in X$ and $K > 0$ such that $d(x, x_0) \leq K$ for all $x \in A$. meaning that $A \subset B_K(x_0)$.

Remark:

i. A is bounded if and only if there exist positive number K such that $d(x, y) \leq K \ \forall x, y \in A$.

ii. A is bounded if and only if $\delta(A) < \infty$.

Def.: let (X, d) be a metric space. A subset A is said to be open set if given any point $x \in A$, there exists $r > 0$ such that $B_r(x) \subseteq A$.

Theorem:

let (X, d) be a metric space. Then

1. Every open ball in metric space X is open set.
2. A subset of X is open iff it is union of a family of open balls.
3. Any finite subset of metric space X is closed.
4. Every metric space is first countable.

Def.: A sequence $\{x_n\}$ in a metric space X is said to be

1. Converge to the point $x \in X$, if for each $\epsilon > 0$, there a positive integer number k such that $d(x_n, x) < \epsilon \ \forall n >$
2. Cauchy sequence if for each $\epsilon > 0$, there is positive integer K such that $d(x_n, x_m) < \epsilon$, $\forall m, n > K$.

Theorem: In a metric space X.

1. limit point of sequence is unique.
2. Every convergence sequence is Cauchy sequence, but the converse not true.

Def.: A metric space X is said to be complet if every Cauchy sequence is converges to the point $x \in X$.

Def.: A sequence $\{f_n\}$ be a sequence of functions from a metric space (X, d_1) into metric space (Y, d_2) is said to be:

i. Converges pointwise to $f: X \to Y$, if every $\epsilon > 0$ there is $K \in \mathbb{Z}^+$ such that $|f_n(x) - f(x)| < \epsilon$, $\forall n > K$.
We write $\lim\limits_{n \to \infty} f_n(x) = f(x)$ or $f_n \to f$ on A.

ii. Uniformly convergent to $f: X \to Y$, if every $\epsilon > 0$ there is $K \in \mathbb{Z}^+$ such that $|f_n(x) - f(x)| < \epsilon$, $\forall n > K$ at $x \in A$.
We write ~~$\lim\limits_{n\to\infty}$~~ $f_n \xrightarrow{u} f$ on A.

It is clear that every uniformly convergent sequence is pointwise convergent, but the converse is not true.

Definition: let (X, d_1) and (Y, d_2) be two metric spaces. A function $f: X \to Y$ is said to be

1. Continuous at $x_0 \in X$ if for every $\epsilon > 0$, there exist $\delta > 0$ such that $d_2(f(x), f(x_0)) < \epsilon$ whenever $d_1(x, x_0) < \delta$.

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2. Sequentially continuous at $x_0 \in X$, if $f(x_n) \longrightarrow f(x)$ in Y whenever $x_n \longrightarrow x$ in X.

A function is said to be continuous (sequentially continuous) iff it is continuous (sequentially continuous) at each point of X.

Def.: let (X, d_1) and (Y, d_2) be two metric spaces. A function $f: X \longrightarrow Y$ is said to be uniformly continuous if for every $\epsilon > 0$, there exist a $\delta > 0$ such that $x, y \in X$, $d_1(x, y) < \delta$ implies $d_2(f(x), f(y)) < \epsilon$.

Example: let $(\mathbb{R}, d)$ be usual metric space, the function $f: \mathbb{R} \longrightarrow \mathbb{R}$, defined by $f(x) = 3x$, $\forall x \in \mathbb{R}$ is uniformly continuous.

Remark: Every uniformly continuous is continuous, but the converse is not true.

Example: let $X = [0,1]$, $Y = \mathbb{R}$, $d_1(x, y) = |x - y|$.

$d_1: [0,1] \longrightarrow \mathbb{R}$, $f(x) = x^2$ is uniformly continuous

$g: (0,1] \longrightarrow \mathbb{R}$, $g(x) = \frac{1}{x}$ is continuous but not uniformly cont.

Normed spaces

Def.: A norm on X is function $\|\cdot\|: X \to \mathbb{R}$ having the following properties:

1. $\|x\| \geq 0$, for all $x \in X$.

2. $\|x\| = 0$ iff $x = 0$

3. $\|\alpha x\| = |\alpha| \, \|x\|$, $\forall x \in X$, $\alpha \in F$.

4. $\|x + y\| \leq \|x\| + \|y\|$, $\forall x, y \in X$.

The linear X over F together with $\|\cdot\|$ is called a normed space and is denoted by $(X, \|\cdot\|)$ or simply X.

Note: 1. A norm $\|\cdot\|$ on linear space X is said to be strictly convex if $\|x + y\| = \|x\| + \|y\|$ only when x and y linearly independent.

2. Every subspace of normed space is also normed space.

Definition: A seminorm on X is a function $S: X \to \mathbb{R}$ having the following:

1. $S(\alpha x) = |\alpha| S(x)$, $\forall x \in X$, $\alpha \in F$.

2. $S(x + y) \leq S(x) + S(y)$, $\forall x, y \in X$

A family F of seminorms on is said to be separating if to each $x \neq 0$ corresponds at least one $S \in F$ with $S(x) \neq 0$.

Theorem:

Suppose S is a seminorm on a vector space X. Then

1. $S(0)=0$. 2. $S(-x)=S(x)$ $\forall x \in X$.

3. $S(x-y)=S(y-x)$ $\forall x,y \in X$

4. $|S(x)-S(y)| \leq S(x-y)$, $\forall x,y \in X$

5. $S(x) \geq 0$, $\forall x \in X$.

6. The $N(S)=\{x \in X ; S(x)=0\}$ is subspace of X.

7. The set $A=\{x \in X ; S(x)<1\}$ is Convex and balanced set.

8. S is a norm if it satisfies the condition $S(x) \neq 0$ if $x \neq 0$.

Proof: (1), (2) and (3) direct from definition

(4) put $x=(x-y)+y \Rightarrow S(x)=S((x-y)+y)$
$\leq S(x-y)+S(y)$

$\Rightarrow S(x)-S(y) \leq S(x-y)$ --- (1)

similarly, we set $y=(y-x)+x$, we obtain

$S(y)-S(x) \leq S(x-y)$ --- (2)

From (1) and (2), we get $|S(x)-S(y)| \leq S(x-y)$

(5) Since $|S(x)-S(y)| \leq S(x-y)$ $\forall x,y \in X$

we set $y=0 \Rightarrow |S(x)| \leq S(x)$

Since $|S(x)| \geq 0 \Rightarrow S(x) \geq 0$, $\forall x \in X$.

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(6) since $S(0)=0 \Rightarrow 0 \in \mathcal{N}(S) \Rightarrow \mathcal{N}(S) \neq \phi$ 28

let $x, y \in \mathcal{N}(S)$ and $\alpha, \beta \in F \Rightarrow S(x)=0, S(y)=0$

Thus, $S(\alpha x+\beta y) \leq S(\alpha x)+S(\beta y)$
$= |\alpha|\, S(x) + |\beta|\, S(y)$
$= |\alpha| \cdot 0 + |\beta| \cdot 0 = 0$

$\Rightarrow \alpha x + \beta y \in \mathcal{N}(S) \Rightarrow \mathcal{N}(S)$ subspace of X.

(7) let $x, y \in A$ and $0 \leq \lambda \leq 1$, then

$S(x) < 1$ and $S(y) < 1$

$S(\lambda x + (1-\lambda) y) \leq S(\lambda x) + S((1-\lambda) y)$
$= |\lambda|\, S(x) + |1-\lambda|\, S(y)$

Since $S(x) < 1 \Rightarrow \lambda S(x) < \lambda$
$S(y) < 1 \Rightarrow (1-\lambda) S(y) < 1-\lambda$

$\Rightarrow S(\lambda x + (1-\lambda) y) \leq \lambda S(x) + (1-\lambda) S(y)$
$< \lambda + (1-\lambda) = 1$

$\Rightarrow \lambda x + (1-\lambda) y \in A \Rightarrow A$ is Convex.

let $\lambda \in F$ with $|\lambda| \leq 1$ and let $x \in \lambda A$

$\Rightarrow x = \lambda a$ where $a \in A \Rightarrow S(a) < 1$

since $S(x) = S(\lambda a) = \lambda S(a)$ and $\lambda < 1$, $S(a) <$

$\Rightarrow \lambda S(a) < 1 \Rightarrow S(x) < 1 \Rightarrow x \in A$

$\Rightarrow \lambda A \subset A \Rightarrow A$ is balanced set.

Theorem: Every normed space is metric space.

Proof: let $(X, \|\cdot\|)$ be a normed space. Define

$d: X \times X \longrightarrow \mathbb{R}$ by $d(x,y) = \|x-y\|$ for $x,y \in X$.

1. let $x,y \in X \Rightarrow x-y \in X$ because X is vector space
$\Rightarrow \|x-y\| \geq 0 \Rightarrow d(x,y) \geq 0$

2. let $x,y \in X$

$d(x,y) = 0 \Leftrightarrow \|x-y\| = 0 \Leftrightarrow x-y=0 \Leftrightarrow x=y$

3. Let $x,y \in X \Rightarrow d(x,y) = \|x-y\| = \|y-x\| = d(y,x)$

4. Let $x,y,z \in X$

$\|x-y\| = \|x-z+z-y\| \leq \|x-z\| + \|z-y\|$

$d(x,y) \leq d(x,z) + d(z,y)$.

If follows that d is a metric on X and this metric is called the metric induced by the normed.

Remark: If $x,y,z \in X$, then

1. $d(x+z, y+z) = d(x,y)$, 2. $\|x\| = d(x,0)$.
3. $d(\alpha x, \alpha y) = |\alpha| \, d(x,y)$.

Def.: let X be a normed space.

1. The open ball with center $x_0 \in X$ and radius $r>0$ denoted by $B_r(x_0)$ and define as

$$B_r(x_0) = \{x \in X, \|x-x_0\| < r\}.$$

and closed ball is $\overline{B}_r(x_0) = \{x \in X, \|x-x_0\| \leq r\}$.

2. A subset A of X is said to be bounded if there exist $k > 0$ such that $\|x\| \leq k$, $\forall x \in A$. 30

3. A sequence $\{x_n\}$ in X is converge to the point $x \in X$, if $\lim_{n \to \infty} \| x_n - x \| = 0$, i.e. $\forall \epsilon > 0$, $\exists k \in \mathbb{Z}^+$;

$\ni \| x_n - x \| < \epsilon \ \forall n \geq k$ and we write $\lim_{n \to \infty} x_n = x$

or $x_n \to x$ as $n \to \infty$.

It follows that $x_n \to x$ iff $\| x_n - x \| \to 0$.

4. Cauchy sequence in X, if for every $\epsilon > 0$, $\exists k \in \mathbb{Z}^+$

$\ni \| x_n - x_m \| < \epsilon \ \forall n, m \geq k$.

5. X is called complete if every cauchy sequence in X is converge to a point of X.

6. X is called a Banach space if it is a complete normed space.

Remark: $B_r(x_0) = x_0 + B_r(0) = x_0 + r B_1(0)$.

Indeed

$$B_r(x_0) = \{ x \in X ; \| x - x_0 \| < r \} = \{ x_0 + y ; \| y \| < r \}$$

$$= x_0 + \{ y ; \| y \| < r \} = x_0 + B_r(0).$$

Also

$$B_r(0) = \{ x \in X ; \| x \| < r \} = \{ x \in X ; \frac{\| x \|}{r} < 1 \}$$

$$= \{ ry ; \| y \| < 1 = r \{ y ; \| y \| < 1 \} = r B_1(0)$$